

Satz 1 (Integralsatz von Green):

Sei $M \subset \mathbb{R}^2$ offene Menge, die sich in endl. viele Standardbereiche zerlegen lässt.

$V = (V_1, V_2)$ stetig diffb. Vektorfeld in Umgebung von $\overline{M} = M \cup \delta M$

$$\Rightarrow \int_M \left(\frac{\partial V_1}{\partial y} - \frac{\partial V_2}{\partial x} \right) d(x, y) = - \int_{\delta M} V_1 dx + V_2 dy$$

Hierbei soll der Rand so durchlaufen werden, dass das Innere immer links liegt.

Beweis:

Beweise Satz für jeden Standardbereich. Allg. Satz folgt durch Aufsummierung.

Also o.B.d.A. M ist Standardbereich.

Wähle folgende explizite Parameterdarstellung von δM

$$\begin{cases} [0, 4] & \leftarrow \mathbb{R}^2 \\ t & \mapsto (x(t), y(t)) \end{cases}$$

mit

$$x(t) = \begin{cases} a + t(b - a) & 0 \leq t \leq 1 \\ b & 1 \leq t \leq 2 \\ b - (t - 2)(b - a) & 2 \leq t \leq 3 \\ a & 3 \leq t \leq 4 \end{cases}$$

$$y(t) = \begin{cases} \varphi(a + t(b - a)) & 0 \leq t \leq 1 \\ \varphi(b) + (t - 1)(\psi(b) - \varphi(b)) & 1 \leq t \leq 2 \\ \varphi(b - (t - 2)(b - a)) & 2 \leq t \leq 3 \\ \varphi(a) - (t - 3)(\psi(a) - \varphi(a)) & 3 \leq t \leq 4 \end{cases}$$

$$\begin{aligned} \int_M \frac{\partial V_1}{\partial y} d(x, y) &\stackrel{\text{Satz 1}}{=} \int_a^b \left(\int_{\varphi(x)}^{\psi(x)} \frac{\partial V_1}{\partial y} dy \right) dx \\ &= \int_a^b (V_1(x, \psi(x)) - V_1(x, \varphi(x))) dx \\ &= - \int_b^a V_1(x, \psi(x)) dx - \int_a^b V_1(x, \varphi(x)) dx \\ &= - \int_2^3 V_1(x(t), y(t)) x'(t) dt - \int_0^1 V_1(x(t), y(t)) x'(t) dt \\ &= - \int_0^4 V_1(x(t), y(t)) x'(t) dt \quad (\text{denn } \int_1^2 V_1(x(t), y(t)) x'(t) dt = 0) \\ &= - \int_{\delta S} V_1(x, y) dx \end{aligned}$$

Analog ist zu zeigen: $-\int_M \frac{\partial V_2}{\partial x} d(x, y) = -\int_{\delta S} V_2 dy$ □

Beispiel 1 :

$$M = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x^2 + y^2 < 1\}$$

$$V_1 = 2xy^3 \quad V_2 = 3x^2y^2$$

$$\int_{\delta M} 2xy^3 dx + 3x^2y^2 dy = - \int_M (6xy^2 - 6xy^2) d(x, y) = 0$$

Beispiel 2 :

M wie oben

$$V_1 = x^4 - y^3 \quad V_2 = x^3 - y^4$$

$$\begin{aligned} \int_{\delta M} (x^4 - y^3) dx + (x^3 - y^4) dy &= - \int_M (-3y^2 - x^2) d(x, y) \\ &= 3 \int_M (x^2 + y^2) d(x, y) \quad (\text{Polarkoord.: } x = r \cos \varphi, y = r \sin \varphi) \\ &= 3 \int_0^1 \left(\int_0^{2\pi} r^2 r dr \right) d\varphi = 6\pi \int_0^1 r^3 dr = \frac{6\pi}{4} = \frac{3}{2}\pi \end{aligned}$$

Satz 2 (Gaußscher Integralsatz):

Sei $M \subset \mathbb{R}^3$ Standardbereich $V = (V_1, V_2, V_3)$ stetig diff'bares Vektorfeld auf Umgebung von \overline{M}

Sei N äußerer Normalenvektor auf δM

$$\Rightarrow \int_M \operatorname{div} V \cdot d(x, y, z) = \int_{\delta M} (V \cdot N) d\sigma$$

$$\operatorname{div} V = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$$

$$M = \bigcup_{(x,y) \in M_z} \{(x, y, z) \in \mathbb{R}^3 \mid \varphi(x, y) \leq z \leq \psi(x, y)\}$$

$$\begin{aligned} \delta M &= \Gamma_\varphi \cup \Gamma_\psi \cup \text{Teil parallel z-Achse} \\ &= \delta M_1 \cup \delta M_2 \cup \delta M_3 \end{aligned}$$

$$\text{Für } p = (x, y, z) \in \Gamma_\psi \text{ ist } N(p) = \frac{(-\psi_x, -\psi_y, 1)}{\sqrt{1+\psi_x^2+\psi_y^2}}$$

$$\text{Für } p = (x, y, z) \in \Gamma_\varphi \text{ ist } N(p) = \frac{(\varphi_x, \varphi_y, -1)}{\sqrt{1+\varphi_x^2+\varphi_y^2}}$$

Beweis: Setze $N(p) = (N_1, N_2, N_3)(p)$

$$\begin{aligned}
 \int_M \frac{\partial V_3}{\partial z}(x, y, z) d(x, y, z) &\stackrel{\text{Satz 1}}{=} \int_{M_z} \left(\int_{\varphi(x, y)}^{\psi(x, y)} \frac{\partial V_3}{\partial z}(x, y, z) dz \right) d(x, y) \\
 &\stackrel{\text{H.D.I.}}{=} \int_{M_z} [V_3(x, y, \psi(x, y)) - V_3(x, y, \varphi(x, y))] d(x, y) \\
 &= \int_{M_z} V_3(x, y, \psi(x, y)) \frac{\sqrt{\psi_x^2 + \psi_y^2 + 1}}{\sqrt{\psi_x^2 + \psi_y^2 + 1}} d(x, y) \\
 &+ \int_{M_z} V_3(x, y, \varphi(x, y)) \frac{\sqrt{\varphi_x^2 + \varphi_y^2 + 1}}{\sqrt{\varphi_x^2 + \varphi_y^2 + 1}} d(x, y) \\
 &\text{Auf dem oberen Rand } \delta M_1 \text{ ist } N_3(p) = \frac{1}{\sqrt{\psi_x^2 + \varphi_y^2 + 1}} \\
 &\text{Auf dem unteren Rand } \delta M_2 \text{ ist } N_3(p) = \frac{-1}{\sqrt{\psi_x^2 + \varphi_y^2 + 1}} \\
 &\text{Zur z-Achse paralleler Teil } \delta M_3 \text{ ist } N_3(p) = 0 \\
 &= \int_{M_z} V_3 \cdot N_3 \cdot \sqrt{\psi_x^2 + \psi_y^2 + 1} d(x, y) \\
 &+ \int_{M_z} V_3 \cdot N_3 \cdot \sqrt{\varphi_x^2 + \varphi_y^2 + 1} d(x, y) \\
 &= \int_{\delta M_1} V_3 N_3 d\sigma + \int_{\delta M_2} V_3 N_3 d\sigma + \int_{\delta M_3} V_3 N_3 d\sigma \\
 &= \int_{\delta M} V_3 \cdot N_3 \cdot d\sigma
 \end{aligned}$$

Analog (wegen M x-projezierbar)

$$\begin{aligned}
 \int_M \frac{\partial V_1}{\partial x} d(x, y, z) &= \int_{\delta M} V_1 \cdot N_1 d\sigma \\
 \int_M \frac{\partial V_2}{\partial x} d(x, y, z) &= \int_{\delta M} V_2 \cdot N_2 d\sigma \\
 \Rightarrow \int_M \operatorname{div} V d(x, y, z) &= \int_M \left(\frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z} \right) d(x, y, z) \\
 &= \int_{\delta M} (V_1 N_1 + V_2 N_2 + V_3 N_3) d\sigma = \int_{\delta M} V \cdot N d\sigma
 \end{aligned}$$

Bemerkung :

$\int_{\delta M} V_i N_i d\sigma$ ist für parametrisierte Flächen definiert.

Man hat für die 3 Summanden verschiedene Parametrisierungen.

\Rightarrow Beweis, dass das Flächenintegral unabhängig von der Parametrisierung ist kommt später.

Beispiel 3 :

Sei $M = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leq x^2 + y^2 < 1, 0 < z < \sqrt{1 - (x^2 + y^2)}\}$
 $V = (V_1, V_2, V_3)$ mit $V_1 = xz^2, V_2 = x^2y - z^3, V_3 = 2xy + y^2z$

$$\begin{aligned} \int_{\delta M} V \cdot N d\sigma &= \int_M \operatorname{div} V d(x, y, z) \\ &= \int_M (z^2 + x^2 + y^2) d(x, y, z) \\ &= \int_0^1 \int_0^{2\pi} \int_0^{\frac{\pi}{2}} r^2 r^2 \cos\theta d\theta dy dr \quad (\text{mit Polarkoord.}) \\ &= \dots \end{aligned}$$

Satz 3 (Integralsatz von Stokes):

Sei $D \subset \mathbb{R}^3$ offen und $V = (V_1, V_2, V_3) : D \rightarrow \mathbb{R}^3$ zweimal stetig diff'bares Vektorfeld.

Sei $M \subset \mathbb{R}^2$ offene Menge, die sich in endl. viele Standardbereiche zerlegen lässt.

Sei $F : M \rightarrow F \subset \bar{F} \subset D \subset \mathbb{R}^3$ eine parametrisierte Fläche

$$\Rightarrow \int_F (\operatorname{rot} V \cdot N) d\sigma = \int_{\partial F} (V_1 dx + V_2 dy + V_3 dz)$$

Bew.:

Sei $\tilde{k} : \begin{cases} [a, b] \rightarrow \partial M \\ t \mapsto (\tilde{u}(t), \tilde{v}(t)) \end{cases}$ Parametrisierung von ∂F

Sei $f(u, v) = (x(u, v), y(u, v), z(u, v))$

Setze zur Abkürzung $\tilde{V}_1(u, v) = V_1(x(u, v), y(u, v), z(u, v))$

$$\begin{aligned} \Rightarrow \int_{\partial F} V_1 dx &= \int_a^b \tilde{V}_1(\tilde{u}(t), \tilde{v}(t)) \frac{d}{dt}(x(\tilde{u}(t), \tilde{v}(t))) dt \\ &\stackrel{\text{Kettenregel}}{=} \int_a^b \tilde{V}_1(\tilde{u}(t), \tilde{v}(t)) \left\{ \frac{\partial x}{\partial u}(\tilde{u}(t), \tilde{v}(t)) \tilde{u}'(t) + \frac{\partial x}{\partial v}(\tilde{u}(t), \tilde{v}(t)) \tilde{v}'(t) \right\} dt \\ &\stackrel{\text{Def. Kurvenint.}}{=} \int_{\partial F} \left\{ \tilde{V}_1(u, v) \frac{\partial x}{\partial u}(u, v) du + \tilde{V}_1(u, v) \frac{\partial x}{\partial v}(u, v) dv \right\} \\ &\stackrel{\text{Green}}{=} \int_M \left\{ \frac{\partial}{\partial u}(\tilde{V}_1(u, v) \frac{\partial x}{\partial v}(u, v)) - \frac{\partial}{\partial v}(\tilde{V}_1(u, v) \frac{\partial x}{\partial u}(u, v)) \right\} d(u, v) \\ &= \int_M \left\{ \frac{\partial}{\partial u} V_1(x(u, v), y(u, v), z(u, v)) \frac{\partial x}{\partial v}(u, v) - \frac{\partial}{\partial v} V_1(x(u, v), y(u, v), z(u, v)) \frac{\partial x}{\partial u}(u, v) \right\} d(u, v) \end{aligned}$$

$$\begin{aligned} \stackrel{\text{Produktregel}}{=} \int_M \left\{ \left(\frac{\partial V_1}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial V_1}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial V_1}{\partial z} \frac{\partial z}{\partial u} \right) \frac{\partial x}{\partial v} + V_1 \frac{\partial^2 x}{\partial u \partial v} - \left(\frac{\partial V_1}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial V_1}{\partial y} \frac{\partial y}{\partial v} + \frac{\partial V_1}{\partial z} \frac{\partial z}{\partial v} \right) \frac{\partial x}{\partial u} - V_1 \frac{\partial^2 x}{\partial v \partial u} \right\} d(u, v) - \\ \left(\frac{\partial x \partial y}{\partial u \partial v} - \frac{\partial x \partial y}{\partial v \partial u} - \frac{\partial x \partial y}{\partial u \partial v} \right) \frac{\partial V_1}{\partial y} + \left(\frac{\partial x \partial z}{\partial v \partial u} - \frac{\partial x \partial z}{\partial u \partial v} \right) \frac{\partial V_1}{\partial z} \right\} d(u, v) = (*) \end{aligned}$$

Es war vergleiche §5:

$$N(u, v) = \frac{(D_x, D_y, D_z)}{\sqrt{D_x^2 + D_y^2 + D_z^2}}(u, v) \text{ mit } D_y = \begin{vmatrix} \frac{\partial z}{\partial u} & \frac{\partial x}{\partial u} \\ \frac{\partial z}{\partial v} & \frac{\partial x}{\partial v} \end{vmatrix}, D_z = \begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \end{vmatrix} \text{ und}$$

$$D = \sqrt{D_x^2 + D_y^2 + D_z^2} = \|f_u \times f_v\|$$

$$\begin{aligned} (*) &= \int_M \left\{ -D_z \frac{\partial V_1}{\partial y} + D_y \frac{\partial V_1}{\partial z} \right\} d(u, v) \\ &= \int_M \left(-\frac{\partial V_1}{\partial y} \frac{D_z}{D} + \frac{\partial V_1}{\partial z} \frac{D_y}{D} \right) \|f_u \times f_v\| d(u, v) \\ &= \int_F \left(-\frac{\partial V_1}{\partial y} \frac{D_z}{D} + \frac{\partial V_1}{\partial z} \frac{D_y}{D} \right) d\sigma \end{aligned}$$

Analog erhält man:

$$\begin{aligned} \int_{\partial F} V_2 dy &= \int_F \left(-\frac{\partial V_2}{\partial z} \frac{D_x}{D} + \frac{\partial V_2}{\partial x} \frac{D_z}{D} \right) d\sigma \\ \int_{\partial F} V_3 dx &= \int_F \left(-\frac{\partial V_3}{\partial x} \frac{D_y}{D} + \frac{\partial V_3}{\partial y} \frac{D_x}{D} \right) d\sigma \end{aligned}$$

Summation ergibt:

$$\begin{aligned} \int_{\partial F} V_1 dx + V_2 dy + V_3 dz &= \int_F \left(\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z} \right) \frac{D_x}{D} + \left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x} \right) \frac{D_y}{D} + \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y} \right) \frac{D_z}{D} \right) d\sigma \\ &= \int_F \operatorname{rot} V \cdot N d\sigma \end{aligned}$$

Beispiel :

$$D = \mathbb{R}^3, V(x, y, z) = (-y, x, 1)$$

F = obere Halbkugel vom Radius 2

$$M = \{(u, v) \in \mathbb{R}^2 \mid u^2 + v^2 \leq 4\}$$

$$f : \begin{cases} M \rightarrow F \subset \mathbb{R}^3 \\ (u, v) \mapsto (x(u, v), y(u, v), z(u, v)) \end{cases} \quad \text{mit} \quad \begin{cases} x(u, v) = u \\ y(u, v) = v \\ z(u, v) = \sqrt{4 - u^2 - v^2} \end{cases}$$

$$\operatorname{rot} V = \left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}, \frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}, \frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y} \right) = (0, 0, 2)$$

$$N = \frac{\left(-\frac{\partial z}{\partial u}, -\frac{\partial z}{\partial v}, 1 \right)}{\sqrt{1 + \left(\frac{\partial z}{\partial u} \right)^2 + \left(\frac{\partial z}{\partial v} \right)^2}}$$

$$\operatorname{rot} V \cdot N = \frac{2}{\sqrt{1 + \left(\frac{\partial z}{\partial u} \right)^2 + \left(\frac{\partial z}{\partial v} \right)^2}} = \frac{2}{\|f_u \times f_v\|}$$

$$\int_F \operatorname{rot} V \cdot N d\sigma = \int_M \frac{2}{\|f_u \times f_v\|} \|f_u \times f_v\| d(u, v) = \int_M 2 d(u, v) = 8\pi$$

Satz von Stokes:

$$\int_{\partial F} (-y dx + x dy + dz) = 8\pi$$